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TOPICS IN THE CALCULUS
OF VARIATIONS

by

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B.Sc. Lunds nation

DEPARTMENT OF MATHEMATICS

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LUND, SWEDEN



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Akademisk avhandling som för avläggande av filosofie dok-
tors examen vid tekniska fakulteten vid universitetet i
Lund kommer att offentligen försvaras på Matematiska
Institutionen, Sal B, fredagen den 31 oktober 1980 kl 10.15.



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Document name
DOCTORAL DISSERTATION

Date of issue
October 1980

CODEN:
LUT FD2/(TFMA-(1002)/1-69/(1980)

Sponsoring organization

Title and subtitle

Topics in the calculus of variations

Abstract

The thesis consists of three papers. We produce a calculus of variations proof of the generalized Gauss-Bonnet theorem using an Euler derivative for multiple integral problems. We also define a linear connection in a principal bundle and use it to generalize a result on the motion of rigid bodies. Finally, we study the second variation, introducing the Jacobi form.

Key words

Classification system and/or index terms (if any)

Supplementary bibliographical information

Language
English

ISSN and key title

ISBN

Recipient's notes

Number of pages
69

Price

Security classification

Distributor by (name and address) Department of Mathematics, Box 725, S-220 07 Lund

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Signature Wilhelm Kremer

Date October 6, 1980

The following papers are summarized in this dissertation:

- [A] Kremer, W.: A variational proof of the Gauss-Bonnet formula, Technical report, Lund, 1980.
- [B] Kremer, W.: A linear connection in a principal bundle, Technical report, Lund, 1980.
- [C] Kremer, W.: The Jacobi form and the second variation, Technical report, Lund, 1980.

The calculus of variations has a long history, beginning with John Bernoulli's work in 1696 on the brachistochrone problem. Euler studied the general problem of finding the maximum or minimum of an integral

$$\int_a^b f(t, x^1(t), \dots, x^n(t), \dot{x}^1(t), \dots, \dot{x}^n(t)) dt,$$

where f is a function of $2n+1$ variables, among all curves $t \mapsto (x^1(t), \dots, x^n(t))$ in a given set. This can be interpreted as a problem on a manifold with local coordinates $(x^1, \dots, x^n)$. A necessary condition for an extremal is

$$\frac{\partial f}{\partial x^j} - \frac{d}{dt} \frac{\partial f}{\partial \dot{x}^j} = 0, \forall j.$$

These are the Euler equations.

Starting in the 1960's, intrinsic versions of the calculus of variations appeared in several books (see [8], [9], and [15]). Peetre introduced in [11] still another approach based on the concept of a

"non-linear" exterior derivative. His Euler derivative is exactly the intrinsic version of the Euler equations. It agrees with the exterior derivative when the integrand is a form.

The formula for the Euler derivative turned out to be especially useful in the case of a manifold with connection ∇ . After extending the Euler derivative to higher order single integral problems in [12], Peetre demonstrated its computational power in [13], achieving, among other things, a simple variational proof of the Gauss-Bonnet formula for a 2-dimensional pseudo-Riemannian manifold.

In [A] we give a new proof of the generalized Gauss-Bonnet theorem. It states that for a $(2m)$ -dimensional orientable Riemannian manifold M with the natural volume element dV ,

$$\int_M K_m dV = \chi(M),$$

where K_m is a certain rational function in the components of the curvature and metric tensors, and $\chi(M)$ is the Euler characteristic of M . Thus we get the equality of a differential geometric quantity on M , with the Euler characteristic, which is a topological invariant. Of course, this accounts for the widespread acquaintance with and continued interest in the theorem.

There are now several proofs of the Gauss-Bonnet formula, the famous intrinsic proof of Chern [6] dating from 1944. The proof in our paper, though technical, is elementary in the sense that, for instance, fiber bundles over M or characteristic classes are not used. Instead it relies on the variation of the integral of a certain Lagrangian over a hypersurface. The variation is computed using an Euler derivative for multiple integral problems, which makes

careful use of the Levi-Civita connection on M . This yields the infinitesimal Gauss-Bonnet theorem. The integration of the infinitesimal result (the transport of the hypersurface) is then hindered only by the topological nature of M .

In paper [B], the emphasis on computational aspects is slight. Here, we are interested in generalizing Arnold's treatment of the motion of a rigid body [3] to the curved case. The variational equations in Arnold's paper describing the motion, and originally derived by Euler in 1765, can also be obtained in a slightly different manner. In this interpretation, use is made of a connection (other than the Levi-Civita) natural for the problem at hand.

Hence, we were first confronted with defining a connection on a principal bundle, which in the case of a group reduces to the one we had initially. Once this obstacle was surmounted, explicit formulas for the torsion and curvature of the connection were derived. Finally, Arnold's paper was generalized.

The computational advantages of the Euler derivative had by now proved themselves in a number of examples. We therefore wanted to see what form the second variation would take in this formalism. The rudimentary aspects of such an investigation are to be found in [C]. The title of the paper derives from the Jacobi equation, which arises naturally in an intrinsic guise - the Jacobi form. Both single and multiple integral (first order) problems are handled. The computation of the second variation of the volume of a minimal surface turns out to be surprisingly painless.

In concluding, I would like to thank professor Jaak Peetre for his help during my work with this thesis. I also want to thank Mrs. Ann-Kristin Ottosson for her excellent typing of [B], and Kurt Liljekvist for his patient printing of all my manuscripts.

References

- [1] Allendoerfer, C.B. and Weil, A.: The Gauss-Bonnet theorem for Riemannian polyhedra. Trans. Amer. Math. Soc 53 (1943), 101-129.
- [2] Armsen, M.: A variational proof of the Gauss-Bonnet formula. Manuscripta Math. 20 (1977), 245-253.
- [3] Arnold, V.: Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits, Ann. Inst. Fourier, XVI, no. 1 (1966), 319-361.
- [4] Avez, A.: Formule de Gauss-Bonnet-Chern en métrique de signature quelconque. C. R. Acad. Sci. Paris 255 (1962), 2049-2051.
- [5] Blaschke, W.: Vorlesungen über Differential Geometrie, II. Springer, Berlin, 1923.
- [6] Chern, S.S.: A simple intrinsic proof of the Gauss-Bonnet formula for closed Riemannian manifolds. Ann. of Math. 45 (1944), 747-752.
- [7] Chern, S.S.: Pseudo-Riemannian geometry and the Gauss-Bonnet formula. An. Acad. Brasil. Ci. 35 (1963), 17-26.
- [8] Hermann, R.: Differential geometry and the calculus of variations. Academic Press, New York, London, 1968.
- [9] Malliavin, P.: Géométrie différentielle intrinsèque. Hermann, Paris, 1972.
- [10] Milnor, J.W.: Lectures on Morse Theory. Princeton University Press, Princeton, New Jersey, 1963.
- [11] Peetre, J.: A "non-linear" exterior derivative and the

foundations of the calculus of variations, Technical report, Lund, 1976.

- [12] Peetre, J.: The Euler derivative, an intrinsic approach to the calculus of variations. Math. Scand. 42 (1978), 313-333.
- [13] Peetre, J.: Further comments on the Euler derivative. Technical report, Lund, 1980.
- [14] Spivak, M.: A comprehensive introduction to differential geometry, Volume IV. Publish or Perish, Inc., Boston, 1975.
- [15] Sternberg, S.: Lectures on differential geometry. Prentice-Hall, Englewood Cliffs, 1964.

